3.1

$$f(x) = \sqrt{x} \qquad f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{(\sqrt{x+\Delta x} - \sqrt{x})(\sqrt{x+\Delta x} + \sqrt{x})}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

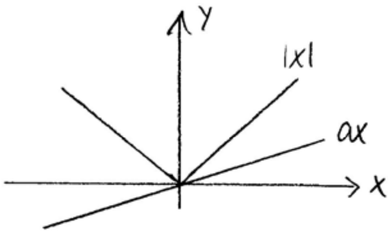
$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

3.2 a)

$$f(x) = |x| \qquad g(x) = ax$$

$$\lim_{x \to x_0} (f(x) - g(x))$$

$$= \lim_{x \to 0} (|x| - ax) = 0$$



b)

$$\lim_{x \to x_0} \frac{f(x) - g(x)}{x - x_0} = \lim_{x \to 0} \frac{|x| - ax}{x} = \lim_{x \to 0} \frac{|x|}{x} - a$$

$$\lim_{\substack{x \to 0 \\ x<0}} \frac{|x|}{x} = \lim_{\substack{x \to 0 \\ x<0}} \frac{-x}{x} = -1 \qquad \lim_{\substack{x \to 0 \\ x>0}} \frac{|x|}{x} = \lim_{\substack{x \to 0 \\ x>0}} \frac{x}{x} = 1$$

$\lim_{x \to 0} \frac{|x|}{x}$ existiert nicht $\qquad \lim_{x \to x_0} \frac{f(x) - g(x)}{x - x_0}$ existiert nicht

Es gibt keine lineare Funktion g mit $g(x_0) = f(x_0)$

und $\lim_{x \to x_0} \frac{f(x) - g(x)}{x - x_0} = 0$

3.3 a)

$f(x) = \sin x \qquad g(x) = |x|$

$(f \circ g)(x) = f(g(x)) = \sin|x|$

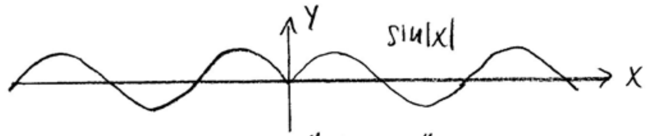
g nicht diff.bar bei $x = 0$

f diff.bar an allen Stellen

Stellen, an denen $f \circ g$ nicht diff.bar sein kann: $x = 0$

$(f \circ g)(x) = \sin x$ für $x \geq 0$

$(f \circ g)(-x) = (f \circ g)(x)$ gerade Funktion



„Knick" / „Spitze" bei $x = 0$

$$(f \circ g)'(x) = \lim_{x \to 0} \frac{(f \circ g)(x) - (f \circ g)(0)}{x - 0} = \lim_{x \to 0} \frac{\sin|x|}{x}$$

$$\lim_{\substack{x \to 0 \\ x < 0}} \frac{\sin|x|}{x} = \lim_{\substack{x \to 0 \\ x < 0}} \frac{\sin(-x)}{x} = -\lim_{\substack{x \to 0 \\ x < 0}} \frac{\sin x}{x} = -1$$

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{\sin|x|}{x} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{\sin x}{x} = 1$$

$\lim_{x \to 0} \frac{\sin|x|}{x}$ existiert nicht

$f \circ g$ nicht diff.bar bei $x = 0$

3.3 b)

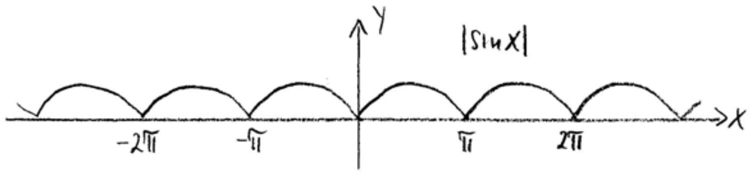
$(g \circ f)(x) = g(f(x)) = |\sin x|$

f diff.bar an allen Stellen

g nicht diff.bar bei $x = 0$

Stellen, an denen $g \circ f$ nicht diff.bar sein kann: Stellen mit $f(x) = 0$

$f(x) = 0 \quad \sin x = 0$ für $x = k\pi \quad k \in \mathbb{Z}$



„Knicke"/„Spitzen" an den Stellen $k\pi \quad k \in \mathbb{Z}$

$$(g \circ f)'(\pi) = \lim_{x \to \pi} \frac{(g \circ f)(x) - (g \circ f)(\pi)}{x - \pi} = \lim_{x \to \pi} \frac{|\sin x|}{x - \pi}$$

$$\lim_{\substack{x \to \pi \\ x < \pi}} \frac{|\sin x|}{x - \pi} = \lim_{\substack{x \to \pi \\ x < \pi}} \frac{\sin x}{x - \pi} \overset{\text{L'H.}}{=} \lim_{\substack{x \to \pi \\ x < \pi}} \frac{\cos x}{1} = -1$$

$$\lim_{\substack{x \to \pi \\ x > \pi}} \frac{|\sin x|}{x - \pi} = \lim_{\substack{x \to \pi \\ x > \pi}} \frac{-\sin x}{x - \pi} \overset{\text{L'H.}}{=} \lim_{\substack{x \to \pi \\ x > \pi}} \frac{-\cos x}{1} = 1$$

$\lim\limits_{x \to \pi} \dfrac{|\sin x|}{x - \pi}$ existiert nicht

$g \circ f$ nicht diff.bar bei $x = \pi$

Ebenso an den anderen Stellen $k\pi \quad k \in \mathbb{Z}$

3.4 a)

$f(x) = \arccos(x^2-1) \qquad D_f = [-\sqrt{2}, \sqrt{2}]$

arccos-Fkt. nicht diff.bar bei $x=-1$ und $x=1$

Stellen, an denen f nicht diff.bar sein kann:

$x^2-1=-1$ oder $x^2-1=1$

$x^2-1=-1 \qquad x=0$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\arccos(x^2-1)-\arccos(-1)}{x}$$

$$= \lim_{x\to 0} \frac{\arccos(x^2-1)-\pi}{x}$$

$$\lim_{\substack{x\to 0\\ x<0}} \frac{\arccos(x^2-1)-\pi}{x} \overset{\text{L'H.}}{=} \lim_{\substack{x\to 0\\ x<0}} \frac{-2x}{\sqrt{1-(x^2-1)^2}}$$

$$= \lim_{\substack{x\to 0\\ x<0}} \frac{-2x}{\sqrt{2x^2-x^4}} = \lim_{\substack{x\to 0\\ x<0}} \frac{-2x}{\sqrt{x^2(2-x^2)}} = \lim_{\substack{x\to 0\\ x<0}} \frac{-2x}{|x|\sqrt{2-x^2}}$$

$$= \lim_{\substack{x\to 0\\ x<0}} \frac{-2x}{-x\sqrt{2-x^2}} = \lim_{\substack{x\to 0\\ x<0}} \frac{2}{\sqrt{2-x^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\lim_{\substack{x\to 0\\ x>0}} \frac{\arccos(x^2-1)-\pi}{x} = \lim_{\substack{x\to 0\\ x>0}} \frac{-2x}{x\sqrt{2-x^2}} = \lim_{\substack{x\to 0\\ x>0}} \frac{-2}{\sqrt{2-x^2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\lim_{x\to 0} \frac{\arccos(x^2-1)-\pi}{x} \quad \text{existiert nicht}$$

f nicht diff.bar bei $x=0$

$x^2 - 1 = 1 \qquad x^2 = 2 \qquad x = \pm\sqrt{2}$

$x = \sqrt{2}$ einseitig (linksseitig) diff.bar bei $x = \sqrt{2}$?

$$f'(\sqrt{2}) = \lim_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{f(x) - f(\sqrt{2})}{x - \sqrt{2}} = \lim_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{\arccos(x^2-1) - \arccos(1)}{x - \sqrt{2}}$$

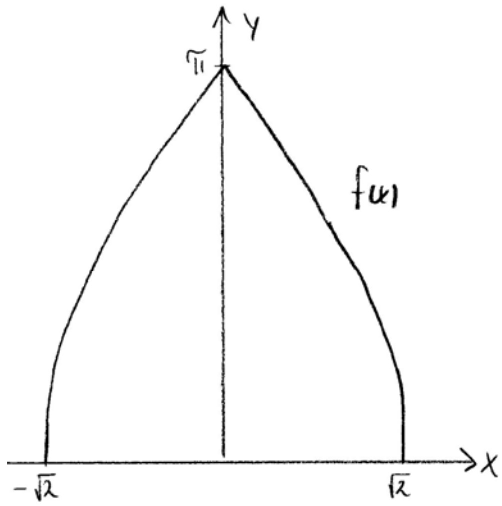
$$= \lim_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{\arccos(x^2-1)}{x - \sqrt{2}} = \lim_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{-2}{\sqrt{2-x^2}} = -\infty$$

$x = -\sqrt{2}$ einseitig (rechtsseitig) diff.bar bei $x = -\sqrt{2}$?

$$f'(-\sqrt{2}) = \lim_{\substack{x \to -\sqrt{2} \\ x > -\sqrt{2}}} \frac{f(x) - f(-\sqrt{2})}{x - (-\sqrt{2})} = \lim_{\substack{x \to -\sqrt{2} \\ x > -\sqrt{2}}} \frac{\arccos(x^2-1)}{x + \sqrt{2}}$$

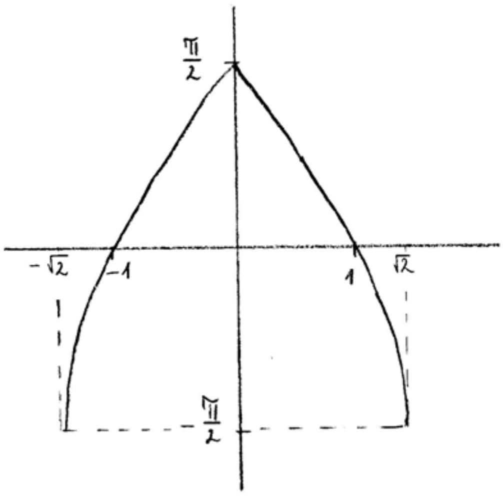
$$= \lim_{\substack{x \to -\sqrt{2} \\ x > -\sqrt{2}}} \frac{2}{\sqrt{2-x^2}} = +\infty$$

f nicht diff.bar an den Stellen $x = \sqrt{2}$ und $x = -\sqrt{2}$



b)

$$f(x) = \arcsin(1-x^2) \qquad D_f = [-\sqrt{2}, \sqrt{2}]$$



arcsin-Fkt. nicht diff.bar bei $x = 1$ und $x = 1$

Stellen, an denen f nicht diff.bar sein kann:

$1 - x^2 = 1 \qquad x = 0$

oder

$1 - x^2 = -1 \qquad x = \pm\sqrt{2}$

$$x = 0 \qquad f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{\arcsin(1-x^2) - \arcsin(1)}{x}$$

$$= \lim_{x \to 0} \frac{\arcsin(1-x^2) - \frac{\pi}{2}}{x}$$

$$\lim_{\substack{x \to 0 \\ x < 0}} \frac{\arcsin(1-x^2) - \frac{\pi}{2}}{x} \overset{\text{L'H.}}{=} \lim_{\substack{x \to 0 \\ x < 0}} \frac{-2x}{\sqrt{1 - (1-x^2)^2}}$$

$$= \lim_{\substack{x \to 0 \\ x < 0}} \frac{-2x}{\sqrt{2x^2 - 4x^4}} = \lim_{\substack{x \to 0 \\ x < 0}} \frac{-2x}{|x|\sqrt{2-x^2}}$$

$$= \lim_{\substack{x \to 0 \\ x < 0}} \frac{-2x}{-x\sqrt{2-x^2}} = \lim_{\substack{x \to 0 \\ x < 0}} \frac{2}{\sqrt{2-x^2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

$$\lim_{\substack{x \to 0 \\ x > 0}} \frac{\arcsin(1-x^2) - \frac{\pi}{2}}{x} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{-2x}{|x|\sqrt{2-x^2}} = \lim_{\substack{x \to 0 \\ x > 0}} \frac{-2x}{x\sqrt{2-x^2}}$$

$$= \lim_{\substack{x \to 0 \\ x > 0}} \frac{-2}{\sqrt{2-x^2}} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

f nicht diff.bar bei $x = 0$

einseitig diff.bar bei $x = \pm\sqrt{2}$

$x = \sqrt{2}$ $\quad f'(\sqrt{2}) = \lim\limits_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{f(x) - f(\sqrt{2})}{x - \sqrt{2}} = \lim\limits_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{\arcsin(1-x^2) - \arcsin(-1)}{x - \sqrt{2}}$

$$= \lim_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{\arcsin(1-x^2) + \frac{\pi}{2}}{x - \sqrt{2}} = \lim_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{-2x}{|x|\sqrt{2-x^2}}$$

$$= \lim_{\substack{x \to \sqrt{2} \\ x < \sqrt{2}}} \frac{-2}{\sqrt{2-x^2}} = -\infty$$

$$f'(-\sqrt{2}) = \lim_{\substack{x \to -\sqrt{2} \\ x > -\sqrt{2}}} \frac{f(x) - f(-\sqrt{2})}{x - (-\sqrt{2})} = \lim_{\substack{x \to -\sqrt{2} \\ x > -\sqrt{2}}} \frac{\arcsin(1-x^2) - \arcsin(-1)}{x + \sqrt{2}}$$

$$= \lim_{\substack{x \to -\sqrt{2} \\ x > -\sqrt{2}}} \frac{\arcsin(1-x^2) + \frac{\pi}{2}}{x + \sqrt{2}} = \lim_{\substack{x \to -\sqrt{2} \\ x > -\sqrt{2}}} \frac{-2x}{|x|\sqrt{2-x^2}}$$

$$= \lim_{\substack{x \to -\sqrt{2} \\ x > -\sqrt{2}}} \frac{2}{\sqrt{2-x^2}} = \infty$$

f nicht (einseitig) diff.bar an den Stellen $x = \pm\sqrt{2}$

c) $f(x) = \sqrt[3]{x^2-2x+1} = \sqrt[3]{(x-1)^2} = (g \circ h)(x)$

mit $g(x) = \sqrt[3]{x}$ und $h(x) = (x-1)^2$

g nicht diff.bar bei $x = 0$

Stellen, an denen f nicht diff.bar sein kann:

Stellen mit $h(x) = 0$ $\quad (x-1)^2 = 0 \quad x = 1$

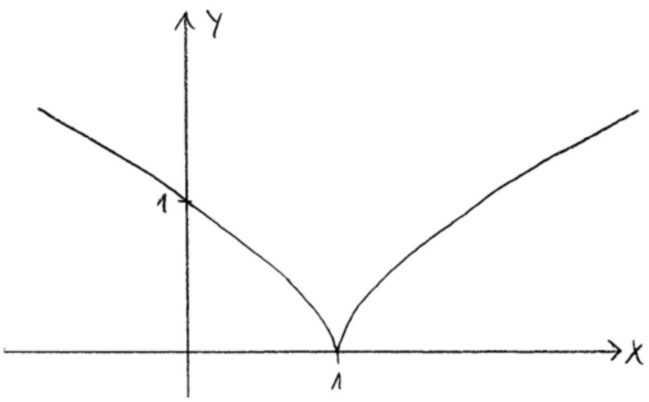
$$f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x-1} = \lim_{x \to 1} \frac{\sqrt[3]{(x-1)^2}}{x-1}$$

$$\lim_{\substack{x \to 1 \\ x > 1}} \frac{\sqrt[3]{(x-1)^2}}{x-1} = \lim_{\substack{x \to 1 \\ x > 1}} \frac{\sqrt[3]{(x-1)^2}}{\sqrt[3]{(x-1)^3}} = \lim_{\substack{x \to 1 \\ x > 1}} \sqrt[3]{\frac{1}{x-1}} = +\infty$$

$$\lim_{\substack{x \to 1 \\ x < 1}} \frac{\sqrt[3]{(x-1)^2}}{x-1} = \lim_{\substack{x \to 1 \\ x < 1}} -\frac{\sqrt[3]{(1-x)^2}}{1-x} = \lim_{\substack{x \to 1 \\ x < 1}} -\frac{\sqrt[3]{(1-x)^2}}{\sqrt[3]{(1-x)^3}}$$

$$= \lim_{\substack{x \to 1 \\ x < 1}} -\sqrt[3]{\frac{1}{1-x}} = -\infty$$

f an nicht diff.bar an der Stelle $x = 1$



3.5 a) $f(x) = \sqrt{1+x^2} \qquad f'(x) = \frac{1}{2\sqrt{1+x^2}} \cdot 2x = \frac{x}{\sqrt{1+x^2}}$

b) $f(x) = x - \sin x \cos x \qquad f'(x) = 1 - \left(\cos x \cdot \cos x + \sin x \cdot (-\sin x)\right)$

$= 1 - \cos^2 x + \sin^2 x = \sin^2 x + \sin^2 x = 2\sin^2 x$

c)

$$f(x) = x\cos(x^2) \qquad f'(x) = \cos(x^2) - x\sin(x^2)2x$$
$$= \cos(x^2) - 2x^2\sin(x^2)$$

d)

$$f(x) = \frac{1}{2}\ln\frac{x^2}{1+x^2} \qquad f'(x) = \frac{1}{2}\ \frac{1+x^2}{x^2}\cdot\frac{2x(1+x^2)-x^2\cdot 2x}{(1+x^2)^2}$$
$$= \frac{1}{2}\ \frac{1+x^2}{x^2}\ \frac{2x}{(1+x^2)^2} = \frac{1}{x(1+x^2)}$$

e)

$$f(x) = 10^{\sqrt{x}} \qquad f'(x) = 10^{\sqrt{x}}\ln 10\ \frac{1}{2\sqrt{x}} = \frac{10^{\sqrt{x}}\ln 10}{2\sqrt{x}}$$

f)

$$f(x) = x^x = e^{x\ln x} \qquad f'(x) = e^{x\ln x}\ (x\ln x)'$$
$$= x^x\left(\ln x + x\frac{1}{x}\right) = x^x(\ln x + 1)$$

g)

$$f(x) = \frac{\ln(x^2+e^x)}{\sqrt{x^2+e^x}}$$

$$f'(x) = \frac{\frac{1}{x^2+e^x}(2x+e^x)\sqrt{x^2+e^x} - \ln(x^2+e^x)\frac{1}{2\sqrt{x^2+e^x}}(2x+e^x)}{x^2+e^x}$$

$$= \frac{2(2x+e^x) - \ln(x^2+e^x)(2x+e^x)}{(x^2+e^x)2\sqrt{x^2+e^x}}$$

$$= \frac{(2x+e^x)(2-\ln(x^2+e^x))}{2\sqrt{x^2+e^x}^{\,3}}$$

h)

$$f(x) = \arctan\frac{e^x}{1+x^2}$$

$$f'(x) = \frac{1}{1+\left(\frac{e^x}{1+x^2}\right)^2}\cdot\frac{e^x(1+x^2)-e^x 2x}{(1+x^2)^2}$$

$$= \frac{1}{1+\frac{e^{2x}}{(1+x^2)^2}}\cdot\frac{e^x(x^2-2x+1)}{(1+x^2)^2}$$

$$= \frac{e^x(x-1)^2}{(1+x^2)^2+e^{2x}}$$

3.6 a)

$$f(x) = \sqrt[3]{(x+1)^2} - \sqrt[3]{(x-1)^2} = \left((x+1)^2\right)^{\frac{1}{3}} - \left((x-1)^2\right)^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}\left((x+1)^2\right)^{-\frac{2}{3}} 2(x+1) - \frac{1}{3}\left((x-1)^2\right)^{-\frac{2}{3}} 2(x-1)$$

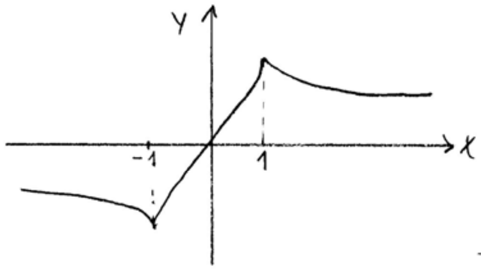
$$= \frac{2}{3}\frac{x+1}{\sqrt[3]{(x+1)^4}} - \frac{2}{3}\frac{x-1}{\sqrt[3]{(x-1)^4}}$$

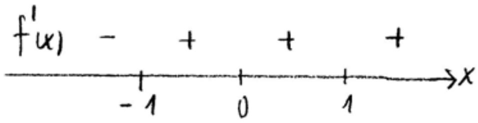
f nicht diff.bar an den Stellen $x = \pm 1$ s. Aufg. 3.4 c)

$$f'(x) = 0 \qquad \frac{x+1}{\sqrt[3]{(x+1)^4}} = \frac{x-1}{\sqrt[3]{(x-1)^4}} \qquad \frac{(x+1)^3}{(x+1)^4} = \frac{(x-1)^3}{(x-1)^4}$$

$$\frac{1}{x+1} = \frac{1}{x-1} \qquad x+1 = x-1 \qquad x = x-2$$

f' hat keine Nullstellen, aber Definitionslücken bei $x = \pm 1$





isol. lokales Minimum bei $x = -1$

isol. lokales Maximum bei $x = 1$

b)

$f(x) = \sqrt{x^2+x^3}$ Wurzelfkt. nicht diff.bar bei $x = 0$

Stellen, an denen f nicht diff.bar sein kann: Stellen mit $x^2+x^3 = 0$

$x^2+x^3 = x^2(1+x) = 0 \qquad x = 0$ oder $x = -1$

$x = 1$ ist Rand/Grenze von $D_f = [-1,\infty[$

$$x = 0 \qquad f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\sqrt{x^2+x^3}}{x} = \lim_{x\to 0} \frac{|x|\sqrt{1+x}}{x}$$

$$\left.\begin{aligned} \lim_{\substack{x\to 0\\ x<0}} \frac{|x|\sqrt{1+x}}{x} &= \lim_{\substack{x\to 0\\ x<0}} \left(-\sqrt{1+x}\right) = -1 \\ \lim_{\substack{x\to 0\\ x>0}} \frac{|x|\sqrt{1+x}}{x} &= \lim_{\substack{x\to 0\\ x>0}} \sqrt{1+x} = 1 \end{aligned}\right\} \Rightarrow$$

$f'(0)$ existiert nicht
f nicht diff.bar
an der Stelle $x = 0$

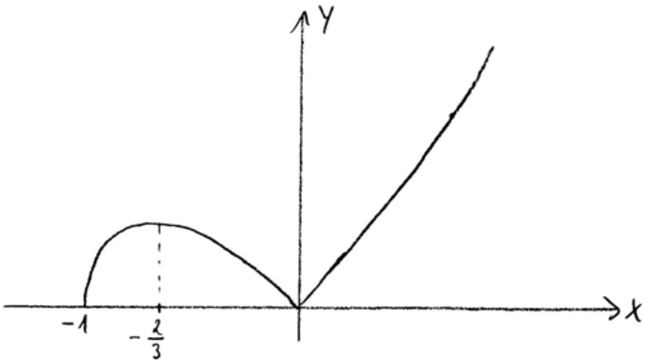
Für $x \neq 0$ gilt $\quad f'(x) = \dfrac{2x+3x^2}{2\sqrt{x^2+x^3}} = \dfrac{x(2+3x)}{2\sqrt{x^2+x^3}}$

$f'(x) = 0 \qquad 2+3x = 0 \qquad x = -\frac{2}{3}$

f' hat Nullstelle bei $x = -\frac{2}{3}$ und Definitionslücke bei $x = 0$

$$\begin{array}{l} f'(x) \quad + \qquad - \qquad + \\ \longrightarrow x \quad (-\tfrac{2}{3},\ 0) \end{array}$$

isol. lokales Maximum bei $x = -\frac{2}{3}$
isol. lokales Minimum bei $x = 0$



3.7

$$f(x) = \begin{cases} x^2 \cos\frac{1}{x} & \text{für } x \neq 0 \\ 0 & \text{für } x = 0 \end{cases}$$

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^2 \cos\frac{1}{x}}{x} = \lim_{x \to 0} x \cos\frac{1}{x}$$

$$= 0 \quad \text{da} \quad \cos\frac{1}{x} \in [-1, 1] \text{ für alle } x \neq 0$$

$$f'(0) = 0$$

3.8 a)

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} x\ln(x^2) = \lim_{x\to 0} \frac{\ln(x^2)}{\frac{1}{x}}$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{\frac{1}{x^2}2x}{-\frac{1}{x^2}} = \lim_{x\to 0}(-2x) = 0$$

f diff.bar bei $x = 0$
f stetig bei $x = 0$

Für $x \neq 0$ gilt $f'(x) = 2x\ln(x^2) + x^2\frac{1}{x^2}2x = 2x\ln(x^2) + 2x$
$= 2x(\ln(x^2)+1)$

$f'(x) = 0 \quad \ln(x^2)+1 = 0 \quad \ln(x^2) = -1 \quad x^2 = e^{-1} = \frac{1}{e}$

$x = \pm\frac{1}{\sqrt{e}}$

$f'(x)$: $-$ | $-\frac{1}{\sqrt{e}}$ | $+$ | 0 | $-$ | $\frac{1}{\sqrt{e}}$ | $+$ → x

isol. lokale Minima bei $x = \pm\frac{1}{\sqrt{e}}$, isol. lokales Maximum bei $x = 0$

$$f''(0) = \lim_{x\to 0}\frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0}\frac{2x(\ln(x^2)+1)}{x} = \lim_{x\to 0}(\ln(x^2)+1) = -\infty$$

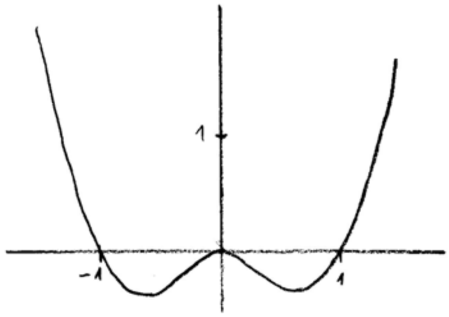
Für $x \neq 0$ gilt $f''(x) = 2\ln(x^2) + 2x\frac{1}{x^2}2x + 2 = 2\ln(x^2)+6$

$f''(x) = 0 \quad 2\ln(x^2)+6 = 0 \quad \ln(x^2) = -3 \quad x^2 = e^{-3} = \frac{1}{e^3} \quad x = \pm\frac{1}{\sqrt{e^3}}$

$f''(x)$ hat Nullstellen bei $x = \pm\frac{1}{\sqrt{e^3}}$ und Definitionslücke bei $x = 0$

$f''(x)$: $+$ | $-\frac{1}{\sqrt{e^3}}$ | $-$ | 0 | $-$ | $\frac{1}{\sqrt{e^3}}$ | $+$ → x

Wendepunkte bei $x = \pm\frac{1}{\sqrt{e^3}}$



b)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} |x|\ln|x|$$

$$\lim_{\substack{x\to 0\\x<0}} |x|\ln|x| = \lim_{\substack{x\to 0\\x<0}} -x\ln(-x) = \lim_{\substack{x\to 0\\x<0}} \frac{\ln(-x)}{-\frac{1}{x}}$$

$$\overset{L'H.}{=} \lim_{\substack{x\to 0\\x<0}} \frac{\frac{1}{x}}{\frac{1}{x^2}} = \lim_{\substack{x\to 0\\x<0}} x = 0$$

ebenso $\lim_{\substack{x\to 0\\x>0}} |x|\ln|x| = \lim_{\substack{x\to 0\\x>0}} x\ln x = 0$

$\Rightarrow \lim_{x\to 0} f(x) = 0 = f(0) \Rightarrow$ f stetig bei $x=0$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{|x|\ln|x|}{x}$$

$\lim_{\substack{x\to 0\\x>0}} \frac{|x|\ln|x|}{x} = \lim_{\substack{x\to 0\\x>0}} \ln x = -\infty \Rightarrow \lim_{x\to 0} \frac{|x|\ln|x|}{x}$ existiert nicht

$\Rightarrow$ f nicht diff.bar an der Stelle $x=0$

Für $x>0$ gilt $f(x) = x\ln x \quad f'(x) = \ln x + 1 \quad f''(x) = \frac{1}{x}$

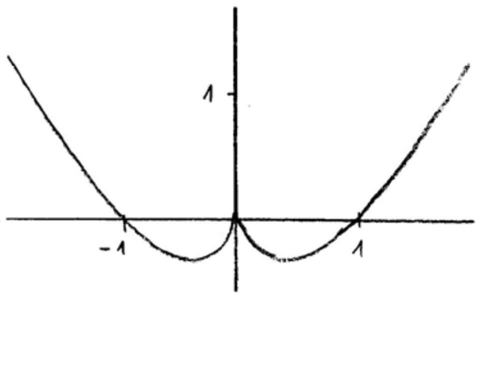
Für $x<0$ gilt $f(x) = -x\ln(-x) \quad f'(x) = -\ln(-x) - 1 \quad f''(x) = -\frac{1}{x}$

$\ln x + 1 = 0 \Rightarrow x = \frac{1}{e} \qquad -\ln(-x) - 1 = 0 \Rightarrow x = -\frac{1}{e}$

$f'(x)$	$-$	$-\frac{1}{e}$	$+$	0	$-$	$\frac{1}{e}$	$+$

drei isolierte lokale Extrema:
Minima bei $x = \pm\frac{1}{e}$
Maximum bei $x=0$

keine Nullstellen von $f''(x)$, keine Wendepunkte



c)

$$\lim_{\substack{x \to 0 \\ x<0}} f(x) = \lim_{\substack{x \to 0 \\ x<0}} \underbrace{\frac{1}{x^3}}_{\to -\infty} \underbrace{e^{-\frac{1}{x}}}_{\to +\infty} = -\infty$$

$\lim\limits_{x \to 0} f(x)$ existiert nicht $\Rightarrow$ f nicht stetig bei $x = 0$

$\Rightarrow$ f nicht diff.bar bei $x = 0$

Für $x \neq 0$ gilt $\quad f'(x) = -\frac{3}{x^4} e^{-\frac{1}{x}} + \frac{1}{x^3} e^{-\frac{1}{x}} \frac{1}{x^2}$

$$= e^{-\frac{1}{x}} \left(\frac{1}{x^5} - \frac{3}{x^4} \right)$$

$f'(x) = 0 \qquad \frac{1}{x^5} - \frac{3}{x^4} = 0 \qquad x = \frac{1}{3}$

$f'(x)$	$-$		$+$		$-$
		0		$\frac{1}{3}$	

isol. lokales Maximum bei $x = \frac{1}{3}$

$$f''(x) = e^{-\frac{1}{x}} \frac{1}{x^2} \left(\frac{1}{x^5} - \frac{3}{x^4} \right) + e^{-\frac{1}{x}} \left(-\frac{5}{x^6} + \frac{12}{x^5} \right)$$

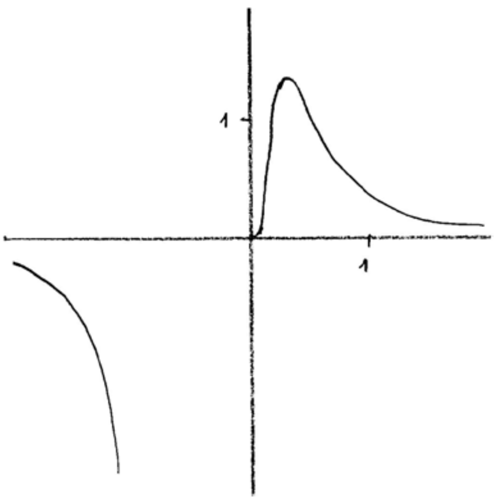
$$= e^{-\frac{1}{x}} \left(\frac{1}{x^7} - \frac{8}{x^6} + \frac{12}{x^5} \right)$$

$f''(x) = 0 \qquad \frac{1}{x^7} - \frac{8}{x^6} + \frac{12}{x^5} = 0 \qquad 12x^2 - 8x + 1 = 0$

$x_1 = \frac{1}{6} \quad x_2 = \frac{1}{2}$

$f''(x)$	$-$		$+$		$-$		$+$
		0		$\frac{1}{6}$		$\frac{1}{2}$	

Wendepunkte bei $x = \frac{1}{6}$ und $x = \frac{1}{2}$



d)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{1}{x^2} e^{-\frac{1}{x^2}} = \lim_{x\to 0} \frac{\frac{1}{x^2}}{e^{\frac{1}{x^2}}} \overset{L'H.}{=} \lim_{x\to 0} \frac{-\frac{2}{x^3}}{e^{\frac{1}{x^2}}\left(-\frac{2}{x^3}\right)}$$

$$= \lim_{x\to 0} \frac{1}{e^{\frac{1}{x^2}}} = 0 = f(0) \quad \Rightarrow \quad f \text{ stetig bei } x=0$$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{\frac{1}{x^2} e^{-\frac{1}{x^2}}}{x} = \lim_{x\to 0} \frac{\frac{1}{x^3}}{e^{\frac{1}{x^2}}}$$

$$\overset{L'H.}{=} \lim_{x\to 0} \frac{-\frac{3}{x^4}}{e^{\frac{1}{x^2}}\left(-\frac{2}{x^3}\right)} = \lim_{x\to 0} \frac{\frac{3}{x}}{2e^{\frac{1}{x^2}}} \overset{L'H.}{=} \lim_{x\to 0} \frac{-\frac{3}{x^2}}{2e^{\frac{1}{x^2}}\left(-\frac{2}{x^3}\right)}$$

$$= \lim_{x\to 0} \frac{3x}{4e^{\frac{1}{x^2}}} = 0 \qquad f \text{ diff.bar bei } x=0 \qquad f'(0)=0$$

Für $x \neq 0$ gilt $f'(x) = -\frac{2}{x^3} e^{-\frac{1}{x^2}} + \frac{1}{x^2} e^{-\frac{1}{x^2}} \frac{2}{x^3}$

$$= \frac{2}{x^5} e^{-\frac{1}{x^2}} - \frac{2}{x^3} e^{-\frac{1}{x^2}} = \frac{2}{x^5} e^{-\frac{1}{x^2}} (1-x^2)$$

$$f'(x) = 0 \qquad 1-x^2 = 0 \qquad x = \pm 1$$

$f'(x)$	$+$		$-$		$+$		$-$
		-1		0		1	

drei isolierte lokale Extrema:
Maxima bei $x = \pm 1$
Minimum bei $x = 0$

$$f''(x) = -\frac{10}{x^6} e^{-\frac{1}{x^2}} + \frac{2}{x^5} e^{-\frac{1}{x^2}} \cdot \frac{2}{x^3} + \frac{6}{x^4} e^{-\frac{1}{x^2}} - \frac{2}{x^3} e^{-\frac{1}{x^2}} \frac{2}{x^3}$$

$$= e^{-\frac{1}{x^2}} \left(\frac{4}{x^8} - \frac{14}{x^6} + \frac{6}{x^4} \right) = \frac{2}{x^8} e^{-\frac{1}{x^2}} (3x^4 - 7x^2 + 2)$$

$$f''(x) = 0 \qquad 3x^4 - 7x^2 + 2 = 0 \qquad u = x^2$$

$$3u^2 - 7u + 2 = 0 \qquad u_1 = \frac{1}{3} \quad u_2 = 2$$

$$x_{1;2} = \pm \frac{1}{\sqrt{3}} \qquad x_{3;4} = \pm\sqrt{2}$$

$$f''(0) = \lim_{x\to 0} \frac{f'(x) - f'(0)}{x-0} = \lim_{x\to 0} \frac{\frac{2}{x^5} e^{-\frac{1}{x^2}} (1-x^2)}{x}$$

$$= \lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^4} \cdot 2(1-x^2)$$

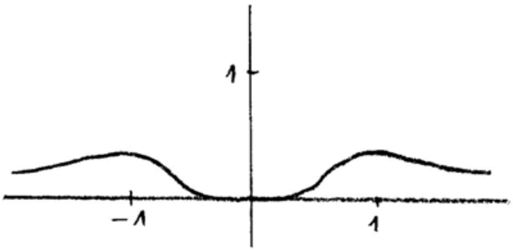
$$\lim_{x\to 0} \frac{e^{-\frac{1}{x^2}}}{x^4} = \lim_{x\to 0} \frac{\frac{1}{x^4}}{e^{\frac{1}{x^2}}} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{-\frac{4}{x^5}}{e^{\frac{1}{x^2}}\left(-\frac{2}{x^3}\right)}$$

$$= \lim_{x\to 0} \frac{\frac{2}{x^2}}{e^{\frac{1}{x^2}}} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{-\frac{4}{x^3}}{e^{\frac{1}{x^2}}\left(-\frac{2}{x^3}\right)} = \lim_{x\to 0} \frac{2}{e^{\frac{1}{x^2}}} = 0$$

5 Nullstellen von $f''(x)$: $0, \pm \frac{1}{\sqrt{3}}, \pm\sqrt{2}$

$f''(x)$	$+$		$-$		$+$		$+$		$-$		$+$
x		$-\sqrt{2}$		$-\frac{1}{\sqrt{3}}$		0		$\frac{1}{\sqrt{3}}$		$\sqrt{2}$	

Wendepunkte bei $x = \pm\sqrt{2}$ und $x = \pm\frac{1}{\sqrt{3}}$



e)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x}{\ln(x^2)} = 0 = f(0) \Rightarrow f \text{ stetig bei } x=0$$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{1}{\ln(x^2)} = 0 \Rightarrow f \text{ diff. bar bei } x=0$$

Für $x \neq 0$ gilt $$f'(x) = \frac{\ln(x^2) - x\,\frac{1}{x^2}\,2x}{(\ln(x^2))^2} = \frac{\ln(x^2)-2}{(\ln(x^2))^2}$$

$$= \frac{1}{\ln(x^2)} - \frac{2}{(\ln(x^2))^2}$$

$f'(x) = 0 \qquad \ln(x^2) - 2 = 0 \qquad \ln(x^2) = 2 \qquad x^2 = e^2 \qquad x = \pm e$

$f'(x)$ hat Nullstellen bei $x = 0, \pm e$

f, f' nicht definiert bei $x = \pm 1$

$f'(x)$	+	$-e$	−	-1	−	0	−	1	−	e	+

2 isolierte lokale Extrema: Maximum bei $x=-e$, Minimum bei $x=e$

$$f''(0) = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0} \frac{1}{x} f'(x)$$

$$= \lim_{x\to 0} \frac{1}{x}\left(\frac{1}{\ln(x^2)} - \frac{2}{(\ln(x^2))^2}\right) = \lim_{x\to 0} \frac{1}{x\ln(x^2)}\left(1 - \frac{2}{\ln(x^2)}\right)$$

$$\lim_{x\to 0} \frac{1}{x\ln(x^2)} = \lim_{x\to 0} \frac{\frac{1}{x}}{\ln(x^2)} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{-\frac{1}{x^2}}{\frac{1}{x^2}\,2x} = \lim_{x\to 0}\left(-\frac{1}{2x}\right) = \pm\infty$$

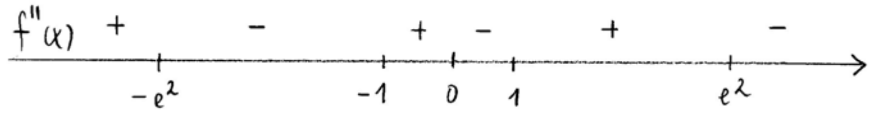
f nicht zweimal diff. bar bei $x=0$

Für $x \neq 0$ gilt: $$f''(x) = -\frac{1}{(\ln(x^2))^2}\,\frac{1}{x^2}\cdot 2x + \frac{4}{(\ln(x^2))^3}\,\frac{1}{x^2}\,2x$$

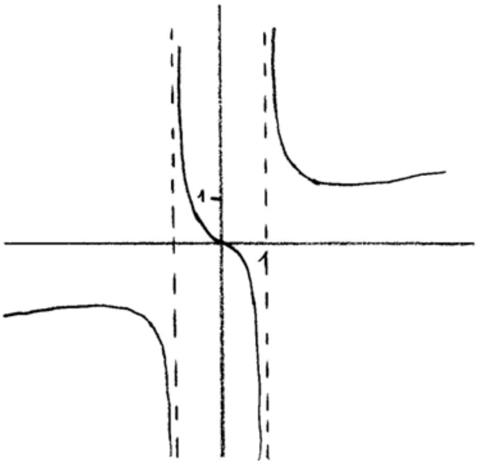
$$= \frac{8}{x(\ln(x^2))^3} - \frac{2}{x(\ln(x^2))^2}$$

$$= \frac{2}{x(\ln(x^2))^3}\left(4 - \ln(x^2)\right)$$

$f''(x) = 0 \qquad 4 - \ln(x^2) = 0 \qquad \ln(x^2) = 4 \qquad x^2 = e^4 \qquad x = \pm e^2$



Wendepunkte bei $x = 0, \pm e^2$



f)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^2}{\ln(x^2)} = 0 = f(0) \Rightarrow f \text{ stetig bei } x = 0$$

$$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x}{\ln(x^2)} = 0 \Rightarrow f \text{ diff.bar bei } x = 0$$

$$x \neq 0: \quad f'(x) = \frac{2x\ln(x^2) - x^2\frac{1}{x^2}2x}{(\ln(x^2))^2} = \frac{2x\ln(x^2) - 2x}{(\ln(x^2))^2}$$

$$= 2x\left(\frac{1}{\ln(x^2)} - \frac{1}{(\ln(x^2))^2}\right)$$

$$f'(x) = 0 \qquad \frac{1}{\ln(x^2)} = \frac{1}{(\ln(x^2))^2} \qquad \ln(x^2) = 1 \qquad x^2 = e \qquad x = \pm\sqrt{e}$$

3 Nullstellen von $f'(x)$: $0, \pm\sqrt{e}$

$f'(x)$	$-$	$-\sqrt{e}$	$+$	-1	$+$	0	$-$	1	$-$	$\sqrt{e}$	$+$

3 isolierte lokale Extrema: Maximum bei $x=0$, Minima bei $x = \pm\sqrt{e}$

$$f''(0) = \lim_{x\to 0} \frac{f'(x)-f'(0)}{x-0} = \lim_{x\to 0} 2\left(\frac{1}{\ln(x^2)} - \frac{1}{(\ln(x^2))^2}\right) = 0$$

$$x \neq 0: \quad f''(x) = 2\left(\frac{1}{\ln(x^2)} - \frac{1}{(\ln(x^2))^2}\right) + 2x\left(-\frac{1}{(\ln(x^2))^2}\frac{1}{x^2}2x + \frac{2}{(\ln(x^2))^3}\frac{1}{x^2}2x\right)$$

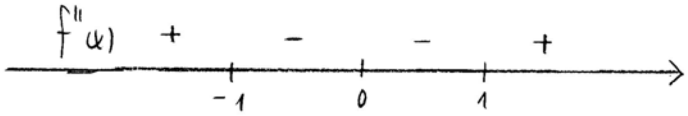
$$= \frac{2}{\ln(x^2)} - \frac{6}{(\ln(x^2))^2} + \frac{8}{(\ln(x^2))^3}$$

$$= \frac{2}{(\ln(x^2))^3}\left((\ln(x^2))^2 - 3\ln(x^2) + 4\right)$$

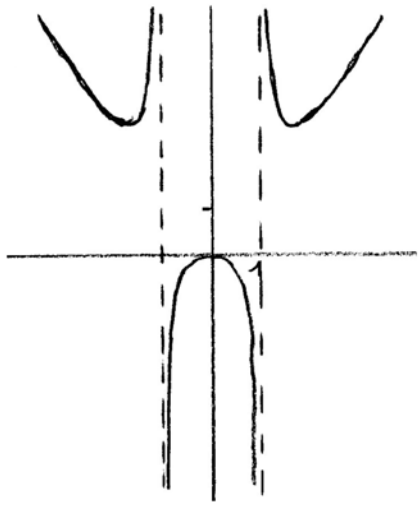
$(\ln(x^2))^2 - 3\ln(x^2) + 4 = 0 \qquad \ln(x^2) = u$

$u^2 - 3u + 4 = 0$ keine Lösung

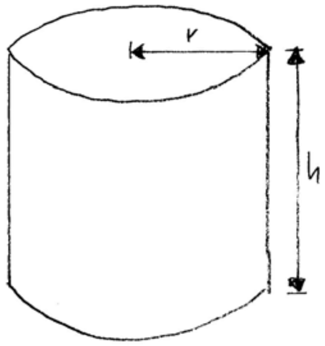
$f''(x) = 0$ fur $x = 0$



kein Wendepunkt



3.9



$V = \pi r^2 h = \frac{1}{2} r \, 2\pi r h$

$A = \pi r^2 + 2\pi r h$

$2\pi r h = A - \pi r^2$

$V = \frac{1}{2} r (A - \pi r^2) = \frac{1}{2} A r - \frac{\pi}{2} r^3$

$f(r) = \frac{1}{2} A r - \frac{\pi}{2} r^3 \qquad f'(r) = \frac{1}{2} A - \frac{3}{2}\pi r^2 = 0 \Rightarrow r = \sqrt{\frac{A}{3\pi}} = r_0$

$f''(r) = -3\pi r \qquad f''(r_0) = -3\pi r_0 < 0$

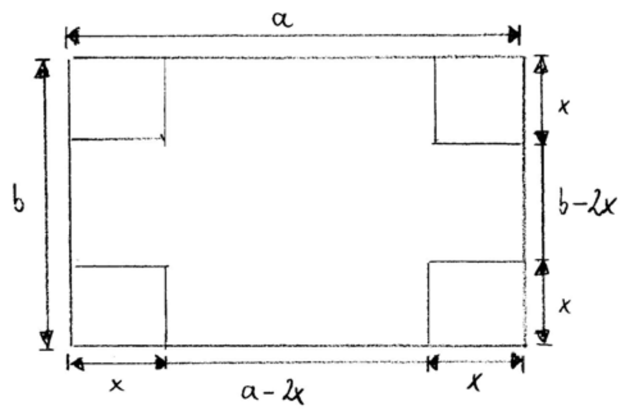
f hat Maximum bei $r = r_0 = \sqrt{\frac{A}{3\pi}}$

$A = \pi r^2 + 2\pi r h \Rightarrow h = \frac{A}{2\pi r} - \frac{1}{2} r = \frac{1}{2} r \left(\frac{A}{\pi r^2} - 1 \right)$

Für $r = r_0$ gilt $h = \frac{1}{2} r_0 \left(\frac{A}{\pi r_0^2} - 1 \right) = \frac{1}{2} r_0 \left(\frac{A}{\pi \frac{A}{3\pi}} - 1 \right) = r_0$

Volumen maximal für $h = r = r_0 = \sqrt{\frac{A}{3\pi}}$

3.10



$$V = (a-2x)(b-2x)x = 4x^3 - 2(a+b)x^2 + abx = f(x)$$

$$f'(x) = 12x^2 - 4(a+b)x + ab$$

$$f'(x) = 0 \Rightarrow x = x_{1;2} = \frac{1}{6}\left(a+b \pm \sqrt{a^2+b^2-ab}\right) \qquad \begin{matrix} + : x_1 \\ - : x_2 \end{matrix}$$

$$f''(x) = 24x - 4(a+b)$$

$$f''(x_1) = 4\sqrt{a^2+b^2-ab} > 0$$

$$f''(x_2) = -4\sqrt{a^2+b^2-ab} < 0$$

f hat Maximum für $x = x_2 = \frac{1}{6}\left(a+b - \sqrt{a^2+b^2-ab}\right)$

Volumen maximal für $x = x_2$

3.11

$$f(x) = \sqrt[3]{x^3+3x^2}$$

$$g(x) = ax+b \quad \text{mit} \quad a = \lim_{x\to\infty} \frac{f(x)}{x} \quad \text{und} \quad b = \lim_{x\to\infty}(f(x)-ax)$$

$$a = \lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \frac{\sqrt[3]{x^3+3x^2}}{x} = \lim_{x\to\infty} \frac{x\sqrt[3]{1+\frac{3}{x}}}{x} = \lim_{x\to\infty} \sqrt[3]{1+\frac{3}{x}} = 1$$

$$b = \lim_{x\to\infty}(f(x)-ax) = \lim_{x\to\infty}\left(\sqrt[3]{x^3+3x^2}-x\right) = \lim_{x\to\infty}\left(x\sqrt[3]{1+\frac{3}{x}}-x\right)$$

$$= \lim_{x\to\infty}\left[x\left(\sqrt[3]{1+\frac{3}{x}}-1\right)\right] = \lim_{x\to\infty} \frac{\left(1+\frac{3}{x}\right)^{\frac{1}{3}}-1}{\frac{1}{x}}$$

$$\overset{L'H.}{=} \lim_{x\to\infty} \frac{\frac{1}{3}\left(1+\frac{3}{x}\right)^{-\frac{2}{3}}\left(-\frac{3}{x^2}\right)}{-\frac{1}{x^2}} = \lim_{x\to\infty} \frac{1}{\sqrt[3]{1+\frac{3}{x}}^{\,2}} = 1$$

$$g(x) = ax+b = x+1$$

3.12 a)

$$f(x) = e^x \qquad x_0 = 0 \qquad f'(x) = e^x$$

$$g(x) = f(x_0) + f'(x_0)(x-x_0) = e^0 + e^0(x-0) = 1+x$$

b)

$$f(x) = \ln x \qquad x_0 = 1 \qquad f'(x) = \frac{1}{x}$$

$$g(x) = f(x_0) + f'(x_0)(x-x_0) = \ln(0) + \frac{1}{1}(x-1) = x-1$$

c)

$$f(x) = \sin x \qquad x_0 = 2\pi \qquad f'(x) = \cos x$$

$$g(x) = f(x_0) + f'(x_0)(x-x_0) = \sin(2\pi) + \cos(2\pi)(x-2\pi) = x-2\pi$$

d)

$$f(x) = 2\sqrt{x^2+3} \qquad x_0 = 1 \qquad f'(x) = \frac{2x}{\sqrt{x^2+3}}$$

$$g(x) = f(x_0) + f'(x_0)(x-x_0) = 2\sqrt{4} + \frac{2}{\sqrt{4}}(x-1) = x+3$$

3.13 a)

$$\lim_{x\to 1} \frac{x+\cos(\pi x)}{x^2+x-2} \overset{L'H.}{=} \lim_{x\to 1} \frac{1-\pi\sin(\pi x)}{2x+1} = \frac{1}{3}$$

b)

$$\lim_{x\to\infty} \frac{\frac{\pi}{2} - \arctan x}{\ln(1+\frac{1}{x})} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{-\frac{1}{1+x^2}}{\frac{1}{1+\frac{1}{x}}\left(-\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{x^2+x}{x^2+1}$$

$$\overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2x}{2x} = 1$$

c)

$$\lim_{\substack{x\to 0\\ x>0}} \left(\frac{1}{x}\right)^x = \lim_{\substack{x\to 0\\ x>0}} e^{x\ln\frac{1}{x}} = \lim_{\substack{x\to 0\\ x>0}} e^{-x\ln x}$$

$$\lim_{\substack{x\to 0\\ x>0}} x\ln x = \lim_{\substack{x\to 0\\ x>0}} \frac{\ln x}{\frac{1}{x}} \overset{\text{L'H.}}{=} \lim_{\substack{x\to 0\\ x>0}} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{\substack{x\to 0\\ x>0}} (-x) = 0$$

$$\Rightarrow \lim_{\substack{x\to 0\\ x>0}} \left(\frac{1}{x}\right)^x = e^0 = 1$$

d)

$$\lim_{x\to 0} \frac{4^x - 2^x}{x} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{4^x\ln 4 - 2^x \ln 2}{1} = \ln 4 - \ln 2 = \ln\frac{4}{2} = \ln 2$$

e)

$$\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right) = \lim_{x\to 1} \frac{x\ln x - (x-1)}{(x-1)\ln x}$$

$$= \lim_{x\to 1} \frac{x\ln x - x + 1}{x\ln x - \ln x} \overset{\text{L'H.}}{=} \lim_{x\to 1} \frac{\ln x + 1 - 1}{\ln x + 1 - \frac{1}{x}} = \lim_{x\to 1} \frac{\ln x}{\ln x + 1 - \frac{1}{x}}$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 1} \frac{\frac{1}{x}}{\frac{1}{x} + \frac{1}{x^2}} = \frac{1}{2}$$

f)

$$\lim_{x\to e} (\ln x)^{\frac{1}{x-e}} = \lim_{x\to e} e^{\frac{1}{x-e}\ln\ln x} = \lim_{x\to e} e^{\frac{\ln\ln x}{x-e}}$$

$$\lim_{x\to e} \frac{\ln\ln x}{x-e} \overset{\text{L'H.}}{=} \lim_{x\to e} \frac{\frac{1}{\ln x}\,\frac{1}{x}}{1} = \frac{1}{e}$$

$$\Rightarrow \lim_{x\to e} (\ln x)^{\frac{1}{x-e}} = e^{\frac{1}{e}}$$

g)

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\lim_{x\to 0}\frac{e^x-1-x}{x(e^x-1)}=\lim_{x\to 0}\frac{e^x-x-1}{xe^x-x}$$

$$\overset{L'H.}{=}\lim_{x\to 0}\frac{e^x-1}{e^x+xe^x-1}\overset{L'H.}{=}\lim_{x\to 0}\frac{e^x}{e^x+e^x+xe^x}=\frac{1}{2}$$

h)

$$\lim_{\substack{x\to 1\\ x>1}}(\ln x)^{\frac{1}{\ln(x-1)}}=\lim_{\substack{x\to 1\\ x>1}}e^{\frac{1}{\ln(x-1)}\ln\ln x}$$

$$\lim_{\substack{x\to 1\\ x>1}}\frac{\ln\ln x}{\ln(x-1)}\overset{L'H.}{=}\lim_{\substack{x\to 1\\ x>1}}\frac{\frac{1}{\ln x}\,\frac{1}{x}}{\frac{1}{x-1}}=\lim_{\substack{x\to 1\\ x>1}}\frac{x-1}{x\ln x}$$

$$\overset{L'H.}{=}\lim_{\substack{x\to 1\\ x>1}}\frac{1}{\ln x+1}=1$$

$$\Rightarrow\lim_{\substack{x\to 1\\ x>1}}(\ln x)^{\frac{1}{\ln(x-1)}}=e$$

i)

$$\lim_{\substack{x\to 0\\ x>0}}\frac{\arcsin(1-x^2)-\frac{\pi}{2}}{x}\overset{L'H.}{=}\lim_{\substack{x\to 0\\ x>0}}\frac{-2x}{\sqrt{1-(1-x^2)^2}}=\lim_{\substack{x\to 0\\ x>0}}\frac{-2x}{\sqrt{2x^2-x^4}}$$

$$=\lim_{\substack{x\to 0\\ x>0}}\frac{-2x}{|x|\sqrt{2-x^2}}=\lim_{\substack{x\to 0\\ x>0}}\frac{-2x}{x\sqrt{2-x^2}}=\lim_{\substack{x\to 0\\ x>0}}\left(-\frac{2}{\sqrt{2-x^2}}\right)$$

$$=-\frac{2}{\sqrt{2}}=-\sqrt{2}$$

j)

$$\lim_{\substack{x\to 0\\ x<0}}\frac{\arccos(x^2-1)-\pi}{x}\overset{L'H.}{=}\lim_{\substack{x\to 0\\ x<0}}\frac{-2x}{\sqrt{1-(x^2-1)^2}}=\lim_{\substack{x\to 0\\ x<0}}\frac{-2x}{\sqrt{2x^2-x^4}}$$

$$=\lim_{\substack{x\to 0\\ x<0}}\frac{-2x}{|x|\sqrt{2-x^2}}=\lim_{\substack{x\to 0\\ x<0}}\frac{-2x}{-x\sqrt{2-x^2}}=\lim_{\substack{x\to 0\\ x<0}}\frac{2}{\sqrt{2-x^2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

k)

$$\ln\cosh x = \ln\left[\frac{1}{2}(e^x + e^{-x})\right] = \ln\frac{1}{2} + \ln(e^x + e^{-x})$$

$$= \ln\left[e^x(1 + e^{-2x})\right] - \ln 2 = \ln(e^x) + \ln(1 + e^{-2x}) - \ln 2$$

$$= x + \ln(1 + e^{-2x}) - \ln 2$$

$$x\tanh x - \ln\cosh x = x\tanh x - x - \ln(1 + e^{-2x}) + \ln 2$$

$$\lim_{x\to\infty}(x\tanh x - x) = \lim_{x\to\infty}\left[x(\tanh x - 1)\right]$$

$$= \lim_{x\to\infty}\frac{\tanh x - 1}{\frac{1}{x}} \overset{\text{L'H.}}{=} \lim_{x\to\infty}\frac{\frac{1}{\cosh^2 x}}{-\frac{1}{x^2}} = -\lim_{x\to\infty}\frac{x^2}{\cosh^2 x}$$

$$\overset{\text{L'H.}}{=} -\lim_{x\to\infty}\frac{2x}{2\cosh x\sinh x} \overset{\text{L'H.}}{=} -\lim_{x\to\infty}\frac{1}{\sinh^2 x + \cosh^2 x} = 0$$

$$\lim_{x\to\infty}(x\tanh x - \ln\cosh x) = \lim_{x\to\infty}\left(x\tanh x - x - \ln(1 + e^{-2x}) + \ln 2\right)$$

$$= \underbrace{\lim_{x\to\infty}(x\tanh x - x)}_{=0} - \underbrace{\lim_{x\to\infty}\ln(1 + e^{-2x})}_{=0} + \ln 2 = \ln 2$$

l)

$$p = \frac{1}{q} \quad \text{mit} \quad q > 1$$

$$\lim_{n\to\infty}(n p^n) = \lim_{n\to\infty}\left(n\left(\frac{1}{q}\right)^n\right) = \lim_{n\to\infty}\frac{n}{q^n} \overset{\text{L'H.}}{=} \lim_{n\to\infty}\frac{1}{nq^{n-1}} = 0$$

3.14 a)

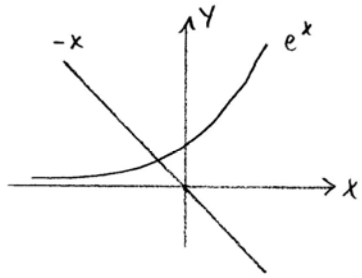
$$f(x) = e^x + x \qquad f'(x) = e^x + 1$$

$$h(x) = x - \frac{f(x)}{f'(x)} = x - \frac{e^x + x}{e^x + 1}$$

$$x_0 = -0{,}5 \quad x_1 = h(x_0) = -0{,}56631$$

$$x_2 = h(x_1) = -0{,}56714$$

$$x_3 = h(x_2) = -0{,}56714$$



b)

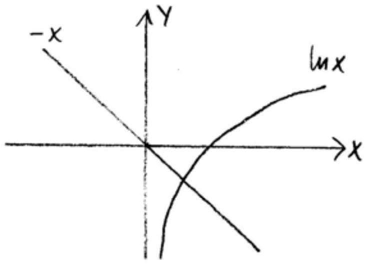
$$f(x) = \ln x + x \qquad f'(x) = \frac{1}{x} + 1$$

$$h(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\ln x + x}{\frac{1}{x} + 1}$$

$x_0 = 0{,}5 \qquad x_1 = h(x_0) = 0{,}56438$

$x_2 = h(x_1) = 0{,}56714$

$x_3 = h(x_2) = 0{,}56714$



c)

$$f(x) = e^{-x} - x^3 \qquad f'(x) = -e^{-x} - 3x^2$$

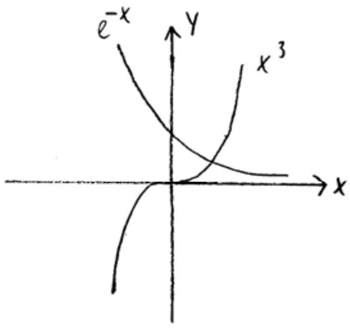
$$h(x) = x - \frac{f(x)}{f'(x)} = x + \frac{e^{-x} - x^3}{e^{-x} + 3x^2}$$

$x_0 = 1 \qquad x_1 = h(x_0) = 0{,}81231$

$x_2 = h(x_1) = 0{,}77428$

$x_3 = h(x_2) = 0{,}77288$

$x_4 = h(x_3) = 0{,}77288$



d)

$$f(x) = \ln x + x^2 \qquad f'(x) = \frac{1}{x} + 2x$$

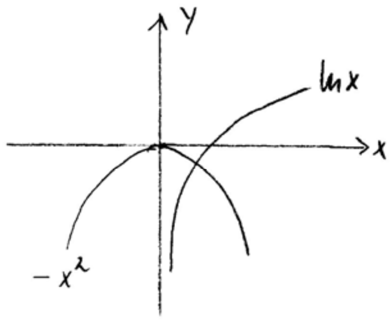
$$h(x) = x - \frac{f(x)}{f'(x)} = x - \frac{\ln x + x^2}{\frac{1}{x} + 2x}$$

$x_0 = 1 \qquad x_1 = h(x_0) = 0{,}66667$

$x_2 = h(x_1) = 0{,}65291$

$x_3 = h(x_2) = 0{,}65292$

$x_4 = h(x_3) = 0{,}65292$



3.15 a)

$$f(x) = \cosh x \qquad f'(x) = \sinh x \qquad f''(x) = \cosh x$$

$$\kappa(x) = \frac{f''(x)}{\sqrt{1 + f'(x)^2}^{\,3}} = \frac{\cosh x}{\sqrt{1 + \sinh^2 x}^{\,3}} = \frac{\cosh x}{\sqrt{\cosh^2 x}^{\,3}} = \frac{\cosh x}{\cosh^3 x}$$

$$= \frac{1}{\cosh^3 x}$$ maximal für $x = 0$

b)

$$f(x) = \operatorname{arcosh} x \qquad f'(x) = \frac{1}{\sqrt{x^2-1}} = (x^2-1)^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{2}(x^2-1)^{-\frac{3}{2}}\, 2x = -\frac{x}{\sqrt{x^2-1}^{\,3}}$$

$$1+f'(x)^2 = 1+\left(\frac{1}{\sqrt{x^2-1}}\right)^2 = 1+\frac{1}{x^2-1} = \frac{x^2-1}{x^2-1}+\frac{1}{x^2-1}$$

$$= \frac{x^2}{x^2-1}$$

$$\sqrt{1+f'(x)^2} = \frac{x}{\sqrt{x^2-1}} \qquad \sqrt{1+f'(x)^2}^{\,3} = \frac{x^3}{\sqrt{x^2-1}^{\,3}}$$

$$\kappa(x) = \frac{f''(x)}{\sqrt{1+f'(x)^2}^{\,3}} = \frac{-\dfrac{x}{\sqrt{x^2-1}^{\,3}}}{\dfrac{x^3}{\sqrt{x^2-1}^{\,3}}} = -\frac{1}{x^2}$$

$|\kappa(x)| = \frac{1}{x^2}$ maximal für $x = 1$

c)

$$f(x) = x^2 \qquad f'(x) = 2x \qquad f''(x) = 2$$

$\kappa(x) = \dfrac{f''(x)}{\sqrt{1+f'(x)^2}^{\,3}} = \dfrac{2}{\sqrt{1+(2x)^2}^{\,3}} = \dfrac{2}{\sqrt{1+4x^2}^{\,3}}$ maximal für $x=0$

d)

$$f(x) = e^x \qquad f'(x) = e^x \qquad f''(x) = e^x$$

$$\kappa(x) = \frac{f''(x)}{\sqrt{1+f'(x)^2}^{\,3}} = \frac{e^x}{\sqrt{1+(e^x)^2}^{\,3}}$$

$$\kappa'(x) = \frac{e^x\sqrt{1+(e^x)^2}^{\,3} - e^x\, 3\sqrt{1+(e^x)^2}\,(e^x)^2}{\left(1+(e^x)^2\right)^3} = 0$$

$$e^x\sqrt{1+(e^x)^2}^{\,3} - e^x\, 3\sqrt{1+(e^x)^2}\,(e^x)^2 = 0$$

$$e^x\sqrt{1+(e^x)^2}\left(1+(e^x)^2-3(e^x)^2\right) = 0$$

$$1-2(e^x)^2 = 0 \qquad (e^x)^2 = \frac{1}{2} \qquad e^x = \frac{1}{\sqrt{2}} \qquad x = \ln\frac{1}{\sqrt{2}} = -\frac{1}{2}\ln 2$$

$\kappa'(x)$ $+$ $-$

$-\frac{1}{2}\ln 2$

$\kappa(x)$ maximal für $x = -\frac{1}{2}\ln 2$

e)

$$f(x) = \ln x \qquad f'(x) = \frac{1}{x} \qquad f''(x) = -\frac{1}{x^2}$$

$$\kappa(x) = \frac{f''(x)}{\sqrt{1+f'(x)^2}^{\,3}} = \frac{-\frac{1}{x^2}}{\sqrt{1+\left(\frac{1}{x}\right)^2}^{\,3}} = -\frac{1}{x^2\sqrt{1+\frac{1}{x^2}}^{\,3}}$$

$$= -\frac{x}{x^3\sqrt{1+\frac{1}{x^2}}^{\,3}} = -\frac{x}{\sqrt{x^2}^{\,3}\sqrt{1+\frac{1}{x^2}}^{\,3}}$$

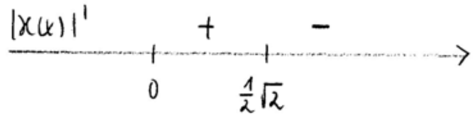
$$= -\frac{x}{\sqrt{x^2\left(1+\frac{1}{x^2}\right)}^{\,3}} = -\frac{x}{\sqrt{x^2+1}^{\,3}}$$

$$|\kappa(x)| = \frac{x}{\sqrt{1+x^2}^{\,3}}$$

$$|\kappa(x)|' = \frac{\sqrt{1+x^2}^{\,3} - 3x^2\sqrt{1+x^2}}{(1+x^2)^3} = 0$$

$$\sqrt{1+x^2}^{\,3} - 3x^2\sqrt{1+x^2} = 0$$

$$\sqrt{1+x^2}\left(1+x^2-3x^2\right) = 0 \qquad 1-2x^2 = 0 \qquad x = \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{2}$$



$|\kappa(x)|$ maximal für $x = \frac{1}{2}\sqrt{2} = \frac{1}{\sqrt{2}}$

f)

$$f(x) = \frac{1}{x} \qquad f'(x) = -\frac{1}{x^2} \qquad f''(x) = \frac{2}{x^3}$$

$$\kappa(x) = \frac{f''(x)}{\sqrt{1+f'(x)^2}^{\,3}} = \frac{\frac{2}{x^3}}{\sqrt{1+\left(\frac{1}{x^2}\right)^2}^{\,3}} = \frac{2}{x^3\sqrt{1+\frac{1}{x^4}}^{\,3}}$$

Fur $x>0$ gilt $x^3 = \sqrt{x^2}^{\,3}$ und

$$\kappa(x) = \frac{2}{\sqrt{x^2}^{\,3}\sqrt{1+\frac{1}{x^4}}^{\,3}} = \frac{2}{\sqrt{x^2\left(1+\frac{1}{x^4}\right)}^{\,3}} = \frac{2}{\sqrt{x^2+\frac{1}{x^2}}^{\,3}}$$

Für $x>0$ ist $\kappa(x)$ maximal, wenn $g(x) = x^2+\frac{1}{x^2}$ minimal ist

$$g(x) = x^2+\frac{1}{x^2} \qquad g'(x) = 2x - \frac{2}{x^3} = 0 \qquad x = 1$$

$g'(x)$		$-$		$+$
	0		1	

g hat Minimum bei $x=1$

$\kappa(x)$ maximal fur $x=1$

Symmetrie von f $\Rightarrow$ Krümmung am stärksten fur $x=\pm 1$

3.16

$$1000\,€\ \frac{q^6-1}{q-1} = 7000\,€ \qquad \frac{q^6-1}{q-1} = 7 \qquad q^6-1 = 7(q-1)$$

$$q^6-1 = 7q-7 \qquad q^6-7q+6 = 0$$

$$f(q) = q^6-7q+6 \qquad f'(q) = 6q^5-7$$

$$h(q) = q - \frac{f(q)}{f'(q)} = q - \frac{q^6-7q+6}{6q^5-7}$$

$q_0 = 1{,}1$

$$q_1 = h(q_0) = 1{,}073128$$
$$q_2 = h(q_1) = 1{,}063154$$
$$q_3 = h(q_2) = 1{,}061453$$
$$q_4 = h(q_3) = 1{,}061402$$
$$q_5 = h(q_4) = 1{,}061402$$

$$q \approx 1{,}0614 \qquad r \approx 0{,}0614 = 6{,}14\,\%$$

3.17

$$d = 2r\cos\frac{\alpha}{2} \qquad A = r^2(\alpha - \sin\alpha) \qquad A = 0{,}67\,\pi r^2$$

$$r^2(\alpha - \sin\alpha) = 0{,}67\,\pi r^2 \qquad \alpha - \sin\alpha = 0{,}67\,\pi$$

$$\alpha - \sin\alpha - 0{,}67\,\pi = 0$$

$$f(\alpha) = \alpha - \sin\alpha - 0{,}67\,\pi \qquad f'(\alpha) = 1 - \cos\alpha$$

$$h(\alpha) = \alpha - \frac{f(\alpha)}{f'(\alpha)} = \alpha - \frac{\alpha - \sin\alpha - 0{,}67\,\pi}{1 - \cos\alpha}$$

$$\alpha_0 = 2 \qquad \alpha_1 = h(\alpha_0) = 2{,}71614$$

$$\alpha_2 = h(\alpha_1) = 2{,}61224$$

$$\alpha_3 = h(\alpha_2) = 2{,}61095$$

$$\alpha_4 = h(\alpha_3) = 2{,}61095$$

$$\alpha = 2{,}61095 \qquad d = 2r\cos\frac{\alpha}{2} = 2{,}6222 \text{ cm}$$

3.18 a)

$$y = f(x) = \frac{v_{y0}}{v_{x0}}x - \frac{g}{2v_{x0}^2}x^2$$

$$f'(x) = \frac{v_{y0}}{v_{x0}} - \frac{g}{v_{x0}^2}x = 0 \qquad x = \frac{v_{x0}v_{y0}}{g} = x_1$$

$$y_{max} = f(x_1) = \frac{v_{y0}}{v_{x0}}\frac{v_{x0}v_{y0}}{g} - \frac{g}{2v_{x0}^2}\left(\frac{v_{x0}v_{y0}}{g}\right)^2 = \frac{v_{y0}^2}{2g}$$

b)

$$W = \frac{2v_0^2}{g}\sin\alpha\cos\alpha = \frac{v_0^2}{g}\sin(2\alpha) = f(\alpha)$$

$$f'(\alpha) = \frac{2v_0^2}{g}\cos(2\alpha) = 0 \Rightarrow 2\alpha = \frac{\pi}{2} \qquad \alpha = \frac{\pi}{4}$$

$$f''(\alpha) = -\frac{4v_0^2}{g}\sin(2\alpha) \qquad f''\left(\frac{\pi}{4}\right) = -\frac{4v_0^2}{g}\sin\frac{\pi}{2} = -\frac{4v_0^2}{g} < 0$$

W maximal für $\alpha = \frac{\pi}{4}$

c)

$$\lim_{\alpha \to \frac{\pi}{2}} \left[\frac{V_0^2}{g} \left(\sin\alpha + \cos^2\alpha \, \operatorname{arsinh}(\tan\alpha) \right) \right]$$

$$= \frac{V_0^2}{g} \left(1 + \lim_{\alpha \to \frac{\pi}{2}} \left(\cos^2\alpha \cdot \operatorname{arsinh}(\tan\alpha) \right) \right)$$

$$\lim \left(\cos^2\alpha \cdot \operatorname{arsinh}(\tan\alpha) \right) = \lim_{\alpha \to \frac{\pi}{2}} \frac{\operatorname{arsinh}(\tanh)}{\frac{1}{\cos^2\alpha}}$$

$$\overset{L'H.}{=} \lim_{\alpha \to \frac{\pi}{2}} \frac{\frac{1}{\sqrt{1+\tanh^2\alpha}} \cdot \frac{1}{\cos^2\alpha}}{2 \frac{\sin\alpha}{\cos^3\alpha}} = \lim_{\alpha \to \frac{\pi}{2}} \frac{\cos\alpha \, \frac{1}{\cos^2\alpha}}{2 \frac{\sin\alpha}{\cos^3\alpha}}$$

$$= \lim_{\alpha \to \frac{\pi}{2}} \frac{\cos^2\alpha}{2\sin\alpha} = 0$$

$$\lim_{\alpha \to \frac{\pi}{2}} \left[\frac{V_0^2}{g} \left(\sin\alpha + \cos^2\alpha \cdot \operatorname{arsinh}(\tan\alpha) \right) \right] = \frac{V_0^2}{g}$$

d)

d) Ges. ist Maximum von $f(\alpha) = \sin\alpha + \cos^2\alpha \,\operatorname{arsinh}(\tan\alpha)$

$$f'(\alpha) = \cos\alpha - 2\cos\alpha\sin\alpha \,\operatorname{arsinh}(\tan\alpha) + \cos^2\alpha \frac{1}{\sqrt{1+\tan^2\alpha}} \frac{1}{\cos^2\alpha}$$

$$= \cos\alpha - 2\cos\alpha\sin\alpha \,\operatorname{arsinh}(\tan\alpha) + \frac{1}{\sqrt{1+\tan^2\alpha}}$$

$$= \cos\alpha - 2\cos\alpha\cdot\sin\alpha \,\operatorname{arsinh}(\tan\alpha) + \cos\alpha$$

$$= 2\cos\alpha\left(1 - \sin\alpha \,\operatorname{arsinh}(\tan\alpha)\right) = 0$$

$$g(\alpha) = \sin\alpha \,\operatorname{arsinh}(\tan\alpha) - 1 = 0$$

$$g'(\alpha) = \cos\alpha \,\operatorname{arsinh}(\tan\alpha) + \sin\alpha \frac{1}{\sqrt{1+\tan^2\alpha}} \frac{1}{\cos^2\alpha}$$

$$= \cos\alpha \,\operatorname{arsinh}(\tan\alpha) + \sin\alpha\cos\alpha \frac{1}{\cos^2\alpha}$$

$$= \cos\alpha \,\operatorname{arsinh}(\tan\alpha) + \tan\alpha$$

$$h(\alpha) = \alpha - \frac{g(\alpha)}{g'(\alpha)} = \alpha - \frac{\sin\alpha \,\operatorname{arsinh}(\tan\alpha) - 1}{\cos\alpha \,\operatorname{arsinh}(\tan\alpha) + \tan\alpha}$$

$\alpha_0 = \frac{\pi}{3} = 1{,}0472$

$\alpha_1 = h(\alpha_0) = 0{,}9884$

$\alpha_2 = h(\alpha_1) = 0{,}9885$

$\alpha_3 = h(\alpha_2) = 0{,}9885$

$\alpha \approx 0{,}9885$ (ca. $56{,}5°$)

$f'(\alpha)$ + −

f hat Maximum bei $\alpha \approx 0{,}9885$

3.19 a)

$$f(x) = \left(V_{y0} + g\frac{m}{k}\right)\frac{x}{V_{x0}} + g\left(\frac{m}{k}\right)^2 \ln\left(1 - \frac{k}{m}\frac{x}{V_{x0}}\right)$$

$$f'(x) = \left(V_{y0} + g\frac{m}{k}\right)\frac{1}{V_{x0}} - g\left(\frac{m}{k}\right)^2 \frac{1}{1-\frac{k}{m}\frac{x}{V_{x0}}}\;\frac{k}{m}\frac{1}{V_{x0}}$$

$$= \left(V_{y0} + g\frac{m}{k}\right)\frac{1}{V_{x0}} - g\frac{m}{k}\frac{1}{1-\frac{k}{m}\frac{x}{V_{x0}}}\;\frac{1}{V_{x0}} = 0$$

$$V_{y0} + g\frac{m}{k} = \frac{g\frac{m}{k}}{1-\frac{k}{m}\frac{x}{V_{x0}}} \qquad 1-\frac{k}{m}\frac{x}{V_{x0}} = \frac{g\frac{m}{k}}{V_{y0}+g\frac{m}{k}} \quad \text{in } f(x)$$

$$\frac{x}{V_{x0}} = \frac{m}{k}\left(1 - \frac{g\frac{m}{k}}{V_{y0}+g\frac{m}{k}}\right) = \frac{m}{k}\,\frac{V_{y0}}{V_{y0}+g\frac{m}{k}} \quad \text{in } f(x)$$

$$h = \left(V_{y0} + g\frac{m}{k}\right)\frac{m}{k}\,\frac{V_{y0}}{V_{y0}+g\frac{m}{k}} + g\left(\frac{m}{k}\right)^2 \ln\frac{g\frac{m}{k}}{V_{y0}+g\frac{m}{k}}$$

$$= \frac{m}{k}V_{y0} - g\left(\frac{m}{k}\right)^2 \ln\frac{V_{y0}+g\frac{m}{k}}{g\frac{m}{k}}$$

$$= \frac{m}{k}V_{y0} - g\left(\frac{m}{k}\right)^2 \ln\left(1 + \frac{V_{y0}}{g\frac{m}{k}}\right)$$

b)

$$h = \frac{m}{k}V_{y0} - g\left(\frac{m}{k}\right)^2 \ln\left(1 + \frac{V_{y0}}{g\frac{m}{k}}\right)$$

$$= \frac{mV_{y0}k - gm^2\ln\left(1 + k\frac{V_{y0}}{mg}\right)}{k^2}$$

$$\lim_{k\to 0} \frac{mV_{y0}k - gm^2\ln\left(1 + k\frac{V_{y0}}{mg}\right)}{k^2}$$

$$\overset{\text{L'H.}}{=} \lim_{k\to 0} \frac{mV_{y0} - gm^2\frac{1}{1+k\frac{V_{y0}}{mg}}\frac{V_{y0}}{mg}}{2k}$$

$$\overset{\text{L'H.}}{=} \lim_{k\to 0} \frac{gm^2\frac{1}{\left(1+k\frac{V_{y0}}{mg}\right)^2}\left(\frac{V_{y0}}{mg}\right)^2}{2} = \frac{{V_{y0}}^2}{2g}$$

c)

$$y = \frac{v_{y0}}{v_{x0}} x + g \frac{m}{k} \frac{x}{v_{x0}} + g\left(\frac{m}{k}\right)^2 \ln\left(1 - \frac{k}{m}\frac{x}{v_{x0}}\right)$$

$$= \frac{v_{y0}}{v_{x0}} x + \frac{gm \frac{x}{v_{x0}} k + gm^2 \ln\left(1 - \frac{k}{m}\frac{x}{v_{x0}}\right)}{k^2}$$

$$\lim_{k\to 0} \frac{gm \frac{x}{v_{x0}} k + gm^2 \ln\left(1 - \frac{k}{m}\frac{x}{v_{x0}}\right)}{k^2}$$

$$\overset{L'H.}{=} \lim_{k\to 0} \frac{gm^2 \frac{x}{v_{x0}} + gm^2 \frac{1}{1-\frac{k}{m}\frac{x}{v_{x0}}} \frac{x}{m v_{x0}}}{2k}$$

$$\overset{L'H.}{=} \lim_{k\to 0} \frac{-gm^2 \frac{1}{\left(1-\frac{k}{m}\frac{x}{v_{x0}}\right)^2}\left(\frac{x}{m v_{x0}}\right)^2}{2} = -\frac{g}{2v_{x0}^2} x^2$$

$$\Rightarrow \quad y \xrightarrow{k\to 0} \frac{v_{y0}}{v_{x0}} x - \frac{g}{2v_{x0}^2} x^2$$

3.20 a)

$$f(v) = \frac{4}{\sqrt{\pi}} a^{\frac{3}{2}} v^2 e^{-av^2} \qquad \text{mit } a = \frac{m}{2kT}$$

$$f'(v) = \frac{4}{\sqrt{\pi}} a^{\frac{3}{2}} \left(2v e^{-av^2} + v^2 e^{-av^2}(-2av)\right)$$

$$= \frac{4}{\sqrt{\pi}} a^{\frac{3}{2}} 2v e^{-av^2}\left(1 - av^2\right) = 0$$

$$1 - av^2 = 0 \qquad v = \sqrt{\frac{1}{a}} = v_1 = \sqrt{\frac{2kT}{m}}$$

$$f'(v) = \frac{8}{\sqrt{\pi}} a^{\frac{3}{2}} v e^{-av^2}(1 - av^2)$$

$$f''(v) = \frac{8}{\sqrt{\pi}} a^{\frac{3}{2}} \left[\left(e^{-av^2} - 2av^2 e^{-av^2}\right)(1-av^2) - 2av^2 e^{-av^2}\right]$$

$$= \frac{8}{\sqrt{\pi}} a^{\frac{3}{2}} e^{-av^2}\left((1-2av^2)(1-av^2) - 2av^2\right)$$

Fur $v = v_1 = \sqrt{\frac{1}{a}}$ ist $1 - av^2 = 0$ $\quad av^2 = 1$ und

$$f''(v_1) = \frac{8}{\sqrt{\pi}} a^{\frac{3}{2}} e^{-1}(-2) = -\frac{16}{\sqrt{\pi}} a^{\frac{3}{2}} \frac{1}{e} < 0$$

$f(v)$ maximal fur $v = v_1 = \sqrt{\frac{1}{a}} = \sqrt{\frac{2kT}{m}}$

b)

$$\frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{\frac{3}{2}} v^2 e^{-\frac{m}{2kT}v^2}$$

$$= b\,T^{-\frac{3}{2}} e^{-\frac{c}{T}} = b\,\frac{T^{-\frac{3}{2}}}{e^{\frac{c}{T}}} \qquad \text{mit } b = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2k}\right)^{\frac{3}{2}} v^2$$

$$\text{und } c = \frac{mv^2}{2k}$$

$$\lim_{T\to 0} \frac{T^{-\frac{3}{2}}}{e^{\frac{c}{T}}} \overset{\text{L'H.}}{=} \lim_{T\to 0} \frac{-\frac{3}{2}T^{-\frac{5}{2}}}{e^{\frac{c}{T}}\left(-\frac{c}{T^2}\right)}$$

$$= \lim_{T\to 0} \frac{\frac{3}{2}T^{-\frac{1}{2}}}{c\,e^{\frac{c}{T}}} \overset{\text{L'H.}}{=} \lim_{T\to 0} \frac{-\frac{1}{2}\frac{3}{2}T^{-\frac{3}{2}}}{c\,e^{\frac{c}{T}}\left(-\frac{c}{T^2}\right)}$$

$$= \lim_{T\to 0} \frac{\frac{3}{4}T^{\frac{1}{2}}}{c^2 e^{\frac{c}{T}}} = \lim_{T\to 0} \frac{3\sqrt{T}}{4c^2 e^{\frac{c}{T}}} = 0$$

3.21 a)

$$C = R\,\frac{x^2 e^x}{(e^x - 1)^2} \qquad \text{mit } x = \frac{T_0}{T}$$

$$T \to 0 \qquad x \to \infty$$

$$\lim_{x\to\infty} \frac{x^2 e^x}{(e^x-1)^2} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2x\,e^x + x^2 e^x}{2(e^x-1)\,e^x} = \lim_{x\to\infty} \frac{2x + x^2}{2(e^x-1)}$$

$$\overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2+2x}{2e^x} \overset{\text{L'H.}}{=} \lim_{x\to\infty} \frac{2}{2e^x} = 0$$

$$C \xrightarrow{T\to 0} 0$$

b)

$$T \to \infty \qquad x = 0$$

$$\lim_{x\to 0} \frac{x^2 e^x}{(e^x-1)^2} \overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{2x\,e^x + x^2 e^x}{2(e^x-1)\,e^x} = \lim_{x\to 0} \frac{2x + x^2}{2(e^x-1)}$$

$$\overset{\text{L'H.}}{=} \lim_{x\to 0} \frac{2+2x}{2e^x} = 1$$

$$C \xrightarrow{T\to\infty} R$$

3.22

$$p = p_0 \left(1 - \frac{a}{x}\right)^x \quad \text{mit} \quad x = \frac{\kappa}{\kappa - 1} \quad \text{und} \quad a = \frac{\varrho_0}{p_0} g h$$

$\kappa \to 1 \qquad x \to \infty$

$$\lim_{x\to\infty} \left(1 - \frac{a}{x}\right)^x = \lim_{x\to\infty} e^{x \ln\left(1 - \frac{a}{x}\right)}$$

$$\lim_{x\to\infty} \left[x \ln\left(1 - \frac{a}{x}\right)\right] = \lim_{x\to\infty} \frac{\ln\left(1 - \frac{a}{x}\right)}{\frac{1}{x}} \stackrel{\text{L'H.}}{=} \lim_{x\to\infty} \frac{\frac{1}{1 - \frac{a}{x}} \cdot \frac{a}{x^2}}{-\frac{1}{x^2}}$$

$$= \lim_{x\to\infty} \left(-\frac{a}{1 - \frac{a}{x}}\right) = -a$$

$$\lim_{x\to\infty} \left(1 - \frac{a}{x}\right)^x = e^{-a}$$

$$p \xrightarrow{\kappa \to 1} p_0 e^{-a} = p_0 e^{-\frac{\varrho_0}{p_0} g h}$$